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Dynamics of a Social Model for Marriage and Divorce Relationship with Fear Effect

Helal, M. M.[1], Yaseen, R. M.[2], Mohsen, A. A.*[1,3], AL-Husseiny, H. F.[4], and Sabbar, Y.[5]

[1]*Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, Iraq*
[2]*Department of Biomedical Engineering, Al-Khawarizmi College of Engineering, University of Baghdad, Iraq*
[3]*Department of Mathematics, Open Educational College, Iraq*
[4]*Department of Mathematics, College of Science, University of Baghdad, Iraq*
[5]*MAIS Laboratory, MAMCS Group, FST Errachidia, Moulay Ismail University of Meknes, P.O. Box 509, Errachidia 52000, Morocco*

E-mail: aamuhseen@gmail.com
**Corresponding author*

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Abstract

The objective of this article is to delve into the intricate dynamics of marriage relationships, exploring the impact of emotions such as fear, love, financial considerations and likability. In our investigation, we adopt a perspective that acknowledges the nonlinear nature of interactions among individuals. Diverging from certain prior studies, we propose that the fear element within the context of marriage is not a singular, isolated factor but rather a manifestation resulting from the amalgamation of numerous social issues. This, in turn, contributes to the emergence of strained and unsuccessful relationships. Unlike conventional approaches, we extensively examine the conditions essential for the existence of all socially significant equilibrium points. A meticulous analysis is undertaken to elucidate the local and global dynamics of the model in the proximity of these equilibrium points. Furthermore, we explore the nuanced interplay between fear, love, money and likability, emphasizing the sensitivity of marriage relationships to changes in the rates of these factors. The outcomes of such variations yield a spectrum of intriguing results within the proposed model, adding depth to our understanding of the complexities inherent in the dynamics of marital relationships.

Keywords: social modeling; marriage; divorce; fear factor; stability analysis; bifurcation analysis.

1 Introduction

The utilization of mathematical models has experienced a notable surge due to the escalating demand for problem-solving across diverse fields such as ecology, biology, chemistry, physics and sociology. Consequently, within the realms of ecology and evolutionary biology over the past few decades, numerous scholars have dedicated their efforts to investigating intricate phenomena such as prey predator interactions and the spread of epidemic diseases. In these contexts, mathematical models have played an instrumental role in significantly advancing our comprehension of the complexities inherent in these challenging scenarios. In the discipline of ecology, the examination of ecological dynamics has become increasingly reliant on mathematical modeling techniques. Researchers have delved into the intricacies of prey-predator interactions, seeking to unveil the underlying principles that govern the relationships between species in ecosystems. Through the employment of sophisticated mathematical models, these studies have not only elucidated the dynamics of predator-prey systems but have also provided valuable insights into the factors influencing population fluctuations and problems that affect the community stability, such as the spread of epidemic diseases. For more information, see: [5], [10], [14]. Similarly, within the domain of evolutionary biology, mathematical modeling has proven indispensable in deciphering the mechanisms that drive the evolution of species over time. The exploration of adaptation, natural selection and genetic drift has been facilitated by the application of mathematical frameworks, enabling researchers to simulate and analyze evolutionary processes with a level of precision that extends beyond traditional observational methods [9]. Moreover, the realm of epidemiology has witnessed a paradigm shift with the increasing reliance on mathematical models to comprehend the dynamics of infectious diseases. The spread and control of epidemics, including factors such as transmission rates, incubation periods and intervention strategies, have been rigorously investigated through mathematical modeling. These models not only contribute to our understanding of disease dynamics but also play a pivotal role in informing public health policies and interventions [15]. In essence, the interdisciplinary application of mathematical models has emerged as an indispensable tool for gaining deeper insights into complex ecological, biological, chemical, physical and social phenomena. The symbiotic relationship between mathematical modeling and scientific inquiry continues to foster a more nuanced and comprehensive understanding of the intricate dynamics that govern the natural world.

In the nascent years of the Nineteenth Century, Gottman purportedly harnessed the power of mathematical models to explicate the intricate patterns inherent in marital relationships. This innovative approach aimed to imbue simulations with a heightened degree of realistic accuracy. The model was meticulously crafted to encapsulate a multifaceted array of factors, including the introduction of a fear factor and the representation of failed relationships. Gottman's pioneering work in this domain marked a paradigm shift, paving the way for a quantitative understanding of the dynamics that underpin the complexities of marital interactions [4]. To delve into the essence of this mathematical modeling endeavor, it is imperative to highlight the incorporation of a fear factor. This parameter reflects the nuanced emotional states within a marriage, encompassing elements such as anxiety, apprehension and emotional distress. By integrating this dimension into the model, Gottman sought to capture the intricate interplay of emotions that can profoundly influence the trajectory of marital relationships. The inclusion of the fear factor elevates the model beyond a simplistic representation, allowing for a more nuanced exploration of the emotional landscape within the context of matrimony [11]. Furthermore, the explicit consideration of failed relationships within the mathematical framework attests to the comprehensive nature of Gottman's approach. By acknowledging and modeling the potential for relationship breakdowns, the model not only reflects the inherent uncertainties within the marital landscape but also serves as a tool for predicting and understanding the factors that contribute to such outcomes. This foresight into the dynamics of failed relationships is invaluable for both theoretical advancements and practical applications, such as counseling and relationship interventions [13], [16]. To elucidate the foundational concept, it is imperative to define marriage as more than a mere social construct. Marriage is a culturally and often legally recognized union between individuals referred to as spouses. It establishes a complex web of rights and obligations not only between the spouses themselves but also extending to encompass their relationships with offspring [17] and [6]. This definition underscores the multifaceted nature of marriage, acknowledging its legal, cultural and familial dimensions. In summary, Gottman's utilization of mathematical models in the analysis of marriage dynamics represents a pioneering venture into the quantitative exploration of complex human relationships. By integrating factors such as the fear factor and the prospect of relationship failure, this approach offers a more nuanced and realistic portrayal of the intricate interplay of emotions and circumstances within the institution of marriage. Moreover, the definition of marriage as a legally and culturally recognized union underscores the intricate web of rights and obligations that bind individuals in marital relationships, emphasizing the interdisciplinary nature of such inquiries.

In pursuit of enhancing marital outcomes, a multitude of scholarly investigations into marriage relationships and associated social phenomena have been undertaken. Ahmed and Khazali [1] have contributed to this discourse by introducing a fractional order love model, seeking to elucidate the dynamics that govern romantic relationships. This fractional order model represents an innovative attempt to capture the nuanced and fractional nature of emotional connections, thereby offering a more refined perspective on the intricacies of love within the context of marriage. In a related vein, Kumar et al. [8] embarked on a mathematical modeling endeavor centered on the legendary love story of Layla and Majnun. By employing mathematical frameworks, they endeavored to dissect the dynamics of this renowned love narrative, providing a unique lens through which to explore the complexities inherent in romantic relationships. This modeling approach not only adds depth to our understanding of love but also showcases the versatility of mathematical tools in interpreting cultural and literary phenomena within the realm of interpersonal dynamics.

Expanding on the exploration of iconic love stories, Jafari et al. [7] delved into the intricacies of the Layla and Majnun saga through a rigorous study. Their work contributes to the interdisciplinary landscape by employing mathematical methodologies to analyze the dynamics of a complex love narrative, shedding light on the patterns and behaviors that characterize such emotionally charged relationships. Shifting the focus to marriage dissolution, Tessema et al.[18] proposed a mathematical model aimed at understanding the dynamics of marriage divorce. This modeling effort not only advances our theoretical grasp of marital challenges but also provides a quantitative framework for exploring the factors that contribute to the breakdown of marital unions. In a predictive vein, Duato and Odar [2] undertook the task of proposing and analyzing a mathematical model for divorce propagation. This model not only allows for the estimation of future divorced populations but also facilitates a systematic exploration of the factors influencing the spread of divorces within a population. Such predictive modeling contributes to a more proactive approach in addressing societal challenges related to marital dissolution. Lastly, Gambrah and Adzadu [3] introduced a nonlinear mathematical model to study the dynamics of divorce epidemics in Ghana. This innovative approach expands the scope of inquiry by considering the nonlinear interactions among married, separated and divorced individuals, providing a comprehensive view of the societal dynamics surrounding divorce. In essence, these scientific endeavors underscore the diverse applications of mathematical modeling in unraveling the intricacies of romantic relationships and marriage. From fractional order love models to analysis of iconic love stories and the dynamics of divorce epidemics, these studies collectively contribute to a richer understanding of the complex interplay of emotions and societal factors within the institution of marriage.

The primary aim of this investigation is to advance our understanding of the dynamics inherent in the social relations of human populations through the development of an innovative mathematical model. This model seeks to provide a more comprehensive depiction of these dynamics by incorporating various influential factors, including fear, love, age and the level of understanding between male and female individuals. The multifaceted nature of human social interactions necessitates a nuanced mathematical framework that accounts for these diverse elements, thereby enriching our ability to model and analyze complex societal dynamics.

The structural framework of this study unfolds as follows: Sections 2 and 3 are dedicated to the formulation of the mathematical model and the exploration of equilibrium points within its dynamics, respectively. In Section 4, a rigorous mathematical analysis is employed to establish stability results for the proposed model. This analytical process allows us to discern the inherent stability or instability of the system, shedding light on the long-term behaviors of the social relations under consideration. Moving beyond theoretical foundations, Section 5 delves into the practical implications of our main results by presenting various applications of the developed model. These applications serve to illustrate the real-world relevance of our mathematical framework, demonstrating its potential utility in interpreting and predicting specific aspects of human social dynamics. By grounding our findings in practical scenarios, we aim to bridge the gap between theoretical abstraction and tangible societal phenomena. Finally, in Section 6, we encapsulate our findings and their implications in a conclusive manner. Section 7 provides a synthesis of the key insights gleaned from the mathematical model, discusses the broader implications of our results and suggests avenues for future research endeavors. By systematically presenting our study in this structured format, we aim to offer a comprehensive exploration of the dynamics of social relations within human populations, providing both theoretical depth and practical applicability to our findings.

2 Model Formulation

In this section, the marriage model is formulated mathematically using a three non-linear ordinary differential equations for describing the model. The model has had three compartments describing how the population distribution of the three species with time. The diagram in Figure 1 and the model equations are as follows:

$$
\begin{aligned}
\frac{dM}{dt} &= rM\left(1-\frac{M}{K}\right)-\frac{\alpha_1 MF}{1+nG}+\frac{\beta G}{2}-\mu M, \qquad M(0)\geq 0,\\
\frac{dF}{dt} &= sF\left(1-\frac{F}{K}\right)-\frac{\alpha_2 MF}{1+nG}+\frac{\beta G}{2}-\mu F, \qquad F(0)\geq 0,\\
\frac{dG}{dt} &= \frac{\alpha_3 MF}{1+nG}-\beta G-\gamma G, \qquad G(0)\geq 0.
\end{aligned} \tag{1}
$$

Here, $M(t)$, $F(t)$ and $G(t)$ represent the densities at time t for the male single individuals, female single individuals and married individuals respectively. It is assumed that males and females grows logistically. Also, it is possible that the marriage is affected by the fear factor. Accordingly, the parameters can be described as in Table 1.

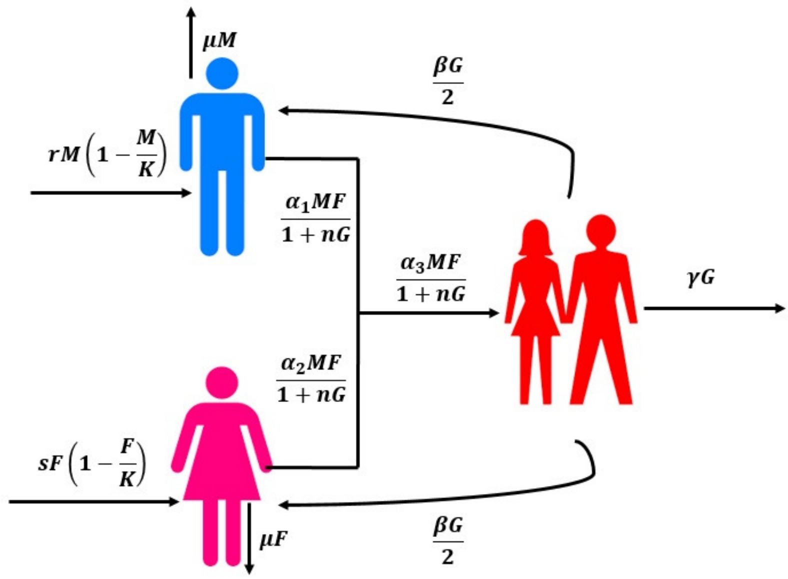

Figure 1: Diagram of marriage model.

Table 1: Definitions of model parameters.

Parameter	Biological Meaning
r	The growth rate of male
s	The growth rate of female
β	The failure marriage rate
K	The carrying capacity
n	The fear rate from marriage
$\alpha_i, i = 1, 2, 3$	The marriage rates with $\alpha_3 \leq min\{\alpha_1, \alpha_2\}$
μ	The death rate
γ	The decay rate
$M(0)$, $F(0)$ and $G(0)$	The initial point

3 The Invariant Region and Equilibrium of Model (1)

Let us determine a region in which the solution of model (1) is bounded. For this model consider a new variable $N = M + F + G$, then by adding all equations of (1), N satisfies:

$$\frac{dN}{dt} = \frac{dM}{dt} + \frac{dN}{dt} + \frac{dG}{dt} = rM\left(1 - \frac{M}{K}\right) + sF\left(1 - \frac{F}{K}\right) - \big(\alpha_1 + \alpha_2 - \alpha_3\big)\frac{MF}{1+nG} - q\big(M + F + G\big),$$

where $q = min\{\mu, \gamma\}$. If there is no marriage, we get

$$\frac{dN}{dt} \leq \frac{(r+s)K}{4} - qN. \tag{2}$$

After solving Equation (2) and evaluating it as $t \to \infty$, we got

$$\Re = \left\{(M,F,G) \in R^3_+ : N(t) \leq \frac{(r+s)K}{4q}\right\},$$

which is the feasible solution set for the model (1) and all the solutions of the model are bounded. Next, the existence of all various equilibrium points (EPs), are considered in following;

- Firstly, if $F = G = 0$, then model (1) has the single males equilibrium point (SMEP), which is denoted $W_1 = (M_1, 0, 0) = ((r-\mu)K/r, 0, 0)$, exits under $\mu < r$ is the biological condition.
- Secondly, if $M = G = 0$, this equilibrium point known as single females equilibrium point (SFEP), which is denoted by $W_2 = (0, F_2, 0) = (0, (s-\mu)K/s, 0)$, exits uniquely under $\mu < s$ is the biological condition.
- Thirdly, if $G = 0$ this equilibrium point known as marriage-free equilibrium point (MFEP) that is denoted by $W_3 = (M_3, F_3, 0)$, exits uniquely under the following conditions:

$$\begin{aligned} rM + \mu K &< rK, \\ rs &< \alpha_1\alpha_2 K^2, \\ rs + \mu\alpha_1 K &< s(\alpha_1 K + \mu), \end{aligned} \tag{3}$$

  where

$$\begin{aligned} M_3 &= \frac{K\Big[s(\alpha_1 K + \mu) - es + \mu\alpha_1 K\Big]}{\alpha_1\alpha_2 K^2 - rs}, \\ F_3 &= \frac{rK - rM_3 - \mu K}{\alpha_1 K}. \end{aligned} \tag{4}$$

- Lastly, model (1) has the marriage equilibrium point (MEP), which is denoted, $W_4 = (M_4, F_4, G_4)$, exits under biological condition (i.e. $\mu < r$) and the following inequality;

$$\begin{aligned} \frac{dM}{dG} &= -\frac{\partial f_1/\partial M}{\partial f_1/\partial G} < 0, \\ \frac{dM}{dG} &= -\frac{\partial f_2/\partial M}{\partial f_2/\partial G} > 0, \end{aligned} \tag{5}$$

  where

$$F_4 = \frac{G_4(1+nG_4)(\beta+\mu)}{\alpha_3 M_4}. \tag{6}$$

  Also, (M_4, G_4) is a positive root to the following two isoclines

$$\begin{aligned} f_1(M_4, G_4) =& 2rM^2 - 2rKM - 2nrKMG + 2nrM^2G + \beta KG + n\beta KG^2 \\ & + 2\mu KM + 2n\mu KMG + \frac{2\alpha_1 KG(1+nG)(\beta+\mu)}{\alpha_3} = 0, \\ f_2(M_4, G_4) =& \beta KG + n\beta KG^2 + \frac{2s(1+nG)^2(\beta G + \mu G)^2}{\alpha_3^2 M^2}(1+nG) \\ & - \frac{2sKG(1+nG)(\beta+\mu)}{\alpha_3 M}(s-\mu) - \frac{2nKG^2(1+nG)(\beta+\mu)}{\alpha_3 M}(s-\mu) \\ & + \frac{2\alpha_2 KG(1+nG)(\beta+\mu)}{\alpha_3} = 0. \end{aligned} \tag{7}$$

Now, if $G \to 0$, Equation (7) can be reduced to

$$\begin{aligned} f_1(M_4, 0) &= 2M(rM + \mu K - rK) = 0, \\ f_2(M_4, 0) &= 0. \end{aligned} \tag{8}$$

Clearly, from the 1^{st} equation of (8), we have a unique positive root that given by

$$M_4 = \frac{K(r-\mu)}{r}, \tag{9}$$

while, the 2^{nd} equation has zero root. Therefore, the all conditions given in Equation (5) with $\mu < r$, guarantees the existence of the marriage equilibrium point (MEP). In the next section, we discussed the local and global stability of all equilibrium points according to calculate the Jacobian matrix.

4 Local and Global Analysis

Theorem 4.1. *If the following conditions are hold, then the (SMEP) W_1 is locally asymptotically stable.*

$$\begin{aligned} rK &< 2rM_1 + \mu K, \\ s &< (\alpha_2 M_1 + \mu), \\ \beta\alpha_3 M_1 &< 2(s - \alpha_2 M_1 - \mu)(\beta + \gamma). \end{aligned} \tag{10}$$

Proof. The Jacobian matrix associated to the model (1) at a given point (M, F, G) can be written as follows:

$$J(M,F,G) = \begin{pmatrix} r - \left(\frac{2rM}{K} + \frac{\alpha_1 F}{1+nG} + \mu\right) & -\frac{\alpha_1 M}{1+nG} & \frac{n\alpha_1 MF}{(1+nG)^2} + \frac{\beta}{2} \\ -\frac{\alpha_2 F}{1+nG} & s - \left(\frac{2sF}{K} + \frac{\alpha_2 M}{1+nG} + \mu\right) & \frac{n\alpha_2 MF}{(1+nG)^2} + \frac{\beta}{2} \\ \frac{\alpha_3 F}{1+nG} & \frac{\alpha_3 M}{1+nG} & -\left(\frac{n\alpha_3 MF}{(1+nG)^2} + \beta + \gamma\right) \end{pmatrix}. \tag{11}$$

Equation (11), evaluated at (SMEP) is given by:

$$J(M_1,0,0) = \begin{pmatrix} r - \left(\frac{2rM_1}{K} + \mu\right) & -\alpha_1 M_1 & \frac{\beta}{2} \\ 0 & s - (\alpha_2 M_1 + \mu) & \frac{\beta}{2} \\ 0 & \alpha_3 M_1 & -(\beta+\gamma) \end{pmatrix}, \tag{12}$$

which admits three distinguish eigenvalues, the first eigenvalue is $\lambda_1 = rK - (2rM_1 + \mu K)$ and two other eigenvalues corresponding to characteristic polynomial is calculated as follows

$$P_1(\lambda) = \lambda^2 + A_1\lambda + A_2,$$

where

$$\begin{aligned} A_1 &= -\big(s - \alpha_2 M_1 - \mu - \beta - \gamma\big), \\ A_2 &= -2\big(s - \alpha_2 M_1 - \mu\big)\big(\beta + \gamma\big) - \beta\alpha_3 M_1. \end{aligned}$$

Thus, if the conditions (10) are hold, then the roots of $P_1(\lambda)$ have negative real parts. Then, (SMEP) is locally asymptotically stable only if conditions (10). Otherwise, it is unstable. □

Theorem 4.2. *If the following conditions are hold, then the (SFEP) W_2 is locally asymptotically stable.*

$$\begin{aligned} sK &< 2sF_2 + \mu K, \\ r &< (\alpha_1 F_2 + \mu), \\ \beta\alpha_3 F_2 &< 2(r - \alpha_1 F_2 - \mu)(\beta + \gamma). \end{aligned} \tag{13}$$

Proof. Equation (11), evaluated at (SFEP) is given by:

$$J(0, F_2, 0) = \begin{pmatrix} r - (\alpha_1 F_2 + \mu) & 0 & \dfrac{\beta}{2} \\ -\alpha_2 F_2 & s\left(1 - \dfrac{2F_2}{K}\right) - \mu & \dfrac{\beta}{2} \\ \alpha_3 F_2 & 0 & -(\beta + \gamma) \end{pmatrix}, \tag{14}$$

which admits three distinguish eigenvalues, the first eigenvalue is $\lambda_1 = sK - (2sF_2 + \mu K)$ and two other eigenvalues corresponding to characteristic polynomial is calculated as follows

$$P_2(\lambda) = \lambda^2 + B_1\lambda + B_2,$$

where

$$\begin{aligned} B_1 &= -(r - \alpha_1 F_2 - \mu - \beta - \gamma), \\ B_2 &= -2(r - \alpha_1 F_2 - \mu)(\beta + \gamma) - \beta\alpha_3 F_2. \end{aligned}$$

Thus, if the conditions (13) are hold, then the roots of $P_2(\lambda)$ have negative real parts. Then, (SFEP) is locally asymptotically stable only if conditions (13). Otherwise, it is unstable. □

Theorem 4.3. *If the following conditions are hold, then the (MFEP) W_3 is locally asymptotically stable.*

$$\begin{aligned} c_{ii} &< 0, \quad i = 1, 2, \\ \mathit{Max}.\{c_{12}c_{21}/c_{22}, \quad c_{13}c_{31}/c_{33}\} &< c_{11} < c_{12}c_{31}/c_{32}, \\ c_{23}c_{32}/c_{33} &< c_{22} < c_{21}c_{32}/c_{31}, \\ c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} &< 2c_{11}c_{22}c_{33}. \end{aligned} \tag{15}$$

Proof. Equation (11), evaluated at (MFEP) is given by:

$$J(M_3, F_3, 0) = [c_{ij}]_{3\times 3}, \tag{16}$$

where

$$\begin{aligned} &c_{11} = r\left(1 - \frac{2M_3}{K}\right) - \alpha_1 F_3 - \mu, \quad c_{12} = -\alpha_1 M_3, \quad c_{13} = n\alpha_1 M_3 F_3 + \frac{\beta}{2}, \\ &c_{21} = -\alpha_2 F_3, \quad c_{22} = s\left(1 - \frac{2F_3}{K}\right) - \alpha_2 M_3 - \mu, \quad c_{23} = \alpha_2 M_3 F_3 + \frac{\beta}{2}, \\ &c_{31} = \alpha_3 F_3, \quad c_{32} = \alpha_3 M_3, \quad c_{33} = -(n\alpha_3 M_3 F_3 + \beta + \gamma). \end{aligned}$$

Now, we have three distinguish eigenvalues, so the corresponding to characteristic polynomial is calculated as follows

$$P_3(\lambda) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3,$$

where

$$\begin{aligned} C_1 =& -\left(c_{11}+c_{22}+c_{33}\right), \\ C_2 =& c_{11}c_{22}+c_{33}\left(c_{11}+c_{22}\right)-\left(c_{12}c_{21}+c_{13}c_{31}+c_{23}c_{32}\right), \\ C_3 =& -c_{33}(c_{11}c_{22}-c_{12}c_{21})+c_{13}(c_{22}c_{31}-c_{21}c_{32})+c_{23}(c_{11}c_{32}-c_{12}c_{31}), \\ C_1C_2-C_3 =& (c_{11}+c_{22})(c_{12}c_{21}-c_{11}c_{22})+(c_{11}+c_{33})(c_{13}c_{31}-c_{11}c_{33}) \\ &+(c_{22}+c_{33})(c_{23}c_{32}-c_{22}c_{33})-2c_{11}c_{22}c_{33}+c_{12}c_{23}c_{31}+c_{13}c_{21}c_{32}. \end{aligned}$$

Thus, if conditions (15) are hold and according to the Routh-Hurwitz criterion conditions (i.e. $C_i > 0; i = 1,3$ and $C_1C_2 - C_3 > 0$), then the roots of $P_3(\lambda)$ have negative real parts. Then, (MFEP) is locally asymptotically stable only if conditions (15). Otherwise, it is unstable. □

Theorem 4.4. *If the following conditions are hold, then the* (*MEP*) W_4 *is locally asymptotically stable.*

$$\begin{aligned} d_{ii} &< 0, \quad i=1,2, \\ \textit{Max.}\ \{d_{12}d_{21}/d_{22},\ \ d_{13}d_{31}/d_{33}\} &< d_{11} < d_{12}d_{31}/d_{32}, \\ d_{23}d_{32}/d_{33} &< d_{22} < d_{21}d_{32}/d_{31}, \\ d_{12}d_{23}d_{31}+d_{13}d_{21}d_{32} &< 2d_{11}d_{22}d_{33}. \end{aligned} \tag{17}$$

Proof. Equation (11), evaluated at (MEP) is given by:

$$J\left(M_4, F_4, G_4\right) = \left[d_{ij}\right]_{3\times 3}, \tag{18}$$

where

$$\begin{aligned} d_{11} &= r\left(1-\frac{2M_4}{K}\right)-\frac{\alpha_1 F_4}{1+nG_4}-\mu, \quad d_{12} = -\frac{\alpha_1 M_4}{1+nG_4}, \quad d_{13} = \frac{n\alpha_1 M_4 F_4}{(1+nG_4)^2}+\frac{\beta}{2}, \\ d_{21} &= -\frac{\alpha_2 F_4}{1+nG_4}, \quad d_{22} = s(1-\frac{2F_4}{K})-\frac{\alpha_2 M_4}{1+nG_4}-\mu, \quad d_{23} = \frac{n\alpha_2 M_4 F_4}{(1+nG_4)^2}+\frac{\beta}{2}, \\ d_{31} &= \frac{\alpha_3 F_4}{1+nG_4}, \quad d_{32} = \frac{\alpha_3 M_4}{1+nG_4}, \quad d_{33} = -(\frac{n\alpha_3 M_4 F_4}{(1+nG_4)^2}+\beta+\gamma). \end{aligned}$$

Now, we have three distinguish eigenvalues, so the corresponding to characteristic polynomial is calculated as follows

$$P_4(\lambda) = \lambda^3 + D_1\lambda^2 + D_2\lambda + D_3,$$

where

$$\begin{aligned} D_1 =& -\left(d_{11}+d_{22}+d_{33}\right), \\ D_2 =& d_{11}d_{22}+d_{33}\left(d_{11}+d_{22}\right)-\left(d_{12}d_{21}+d_{13}d_{31}+d_{23}d_{32}\right), \\ D_3 =& -d_{33}(d_{11}d_{22}-d_{12}d_{21})+d_{13}(d_{22}d_{31}-d_{21}d_{32})+d_{23}(d_{11}d_{32}-d_{12}d_{31}), \\ D_1D_2-D_3 =& (d_{11}+d_{22})(d_{12}d_{21}-d_{11}d_{22})+(d_{11}+d_{33})(d_{13}d_{31}-d_{11}d_{33}) \\ &+(d_{22}+d_{33})(d_{23}d_{32}-d_{22}d_{33})-2d_{11}d_{22}d_{33}+d_{12}d_{23}d_{31}+d_{13}d_{21}d_{32}. \end{aligned}$$

Thus, if conditions (16) are hold and according to the Routh-Hurwitz criterion conditions (i.e. $D_i > 0; i = 1,3$ and $D_1D_2 - D_3 > 0$), then the roots of $P_4(\lambda)$ have negative real parts. Then, (MEP) is locally asymptotically stable only if conditions (16). Otherwise, it is unstable. □

Next, the following theorems are interested with the model's global dynamics (1). The attractive basin of trajectories of a dynamical model (1), according to global stability, is either the state-space or the interior of the state-space that determines the system's state variables. In other words, global stability implies that, regardless of the initial conditions, all paths eventually drift to the system's attractor. Global stability is required by most biological, ecological or social systems.

Theorem 4.5. *The SMEP is globally asymptotically stable under the following requirement holds.*

$$\begin{aligned} \alpha_3 + \alpha_1 M_1 &< \alpha_2, \\ 2\beta G + sK &< 4\mu F, \\ rK &< r(M + M_1) + \mu. \end{aligned} \tag{19}$$

Proof. Let define the function $V_1(M, F, G) = \dfrac{(M - M_1)^2}{2} + F + G$, which is a positive definite real valued function on the region $\Omega_1 = (M, F, G) \in R^3_+ : M > 0, F \geq 0, G \geq 0$. Then, after simplify it by some direct calculations, we have

$$\begin{aligned} \frac{dV_1}{dt} =& - \left[\frac{r}{K}(M + M_1) - (r - \mu)\right](M - M_1)^2 - \left[\frac{\alpha_2 - (\alpha_3 + \alpha_1 M_1)}{1 + nG}\right] MF - \mu F \\ & - (M - M_1)\frac{\beta G}{2} + (1 - \frac{F}{K})sF - \left[(\mu + \beta + \gamma) - \frac{\beta}{2}\right] G. \end{aligned}$$

Further simplification leads to:

$$\begin{aligned} \frac{dV_1}{dt} \leq & - \left[\frac{r}{K}(M + M_1) - (r - \mu)\right](M - M_1)^2 - \left[\frac{\alpha_2 - (\alpha_3 + \alpha_1 M_1)}{1 + nG}\right] MF - \mu F \\ & + \frac{\beta G M_1}{2} + \frac{sK}{4} - \left[(\mu + \beta + \gamma) - \frac{\beta}{2}\right] G. \end{aligned}$$

Clearly, dV_1/dt is negative definite under conditions (19). Hence, the SMEP is GAS. □

Theorem 4.6. *The SFEP is globally asymptotically stable under the following requirement holds.*

$$\begin{aligned} \alpha_3 + \alpha_2 F_2 &< \alpha_1, \\ rK + 2\beta G F_2 &< 4\mu M, \\ sK &< s(F + F_2) + \mu. \end{aligned} \tag{20}$$

Proof. Let define the function $V_2(M, F, G) = M + \frac{(F - F_2)^2}{2} + G$, which is a positive definite real valued function on the region $\Omega_2 = (M, F, G) \in R^3_+ : M \geq 0, F > 0, G \geq 0$. Then, after simplify it by some direct calculations, we have

$$\begin{aligned} \frac{dV_2}{dt} =& rM(1 - \frac{M}{K}) - \left[\frac{\alpha_1 - (\alpha_3 + \alpha_2 F_2)}{1 + nG}\right] MF - \mu M - (F - F_2)\frac{\beta G}{2} \\ & - \left[\frac{s}{K}(F + F_2) + \mu - s\right](F - F_2)^2 - \left[(\beta + \gamma) - \frac{\beta}{2}\right] G. \end{aligned}$$

Further simplification leads to:

$$\begin{aligned} \frac{dV_2}{dt} \leq & \frac{rK}{4} - \left[\frac{\alpha_1 - (\alpha_3 + \alpha_2 F_2)}{1 + nG}\right] MF - \mu M + \frac{\beta G F_2}{2} \\ & - \left[\frac{s}{K}(F + F_2) + \mu - s\right](F - F_2)^2 - \left[(\beta + \gamma) - \frac{\beta}{2}\right] G. \end{aligned}$$

Now, we have dV_2/dt is negative definite under conditions (20). Hence, the SFEP is GAS. □

Theorem 4.7. *The MFEP is globally asymptotically stable under the following requirement holds.*

$$
\begin{aligned}
q_{11} &< 0,\\
q_{22} &< 0,\\
q_{12}^2 &< 4q_{11}q_{22},\\
Max.\ \{q_1^\star, q_2^\star\} &< q_3^\star.
\end{aligned} \tag{21}
$$

Proof. Let define the function $V_3(M, F, G) = \dfrac{(M - M_3)^2}{2} + \dfrac{(F - F_3)^2}{2} + G$, which is a positive definite real valued function on the region $\Omega_3 = (M, F, G) \in R_+^3 : M > 0, F > 0, G \geq 0$. Then, after simplify it by some direct calculations, we have

$$
\begin{aligned}
\frac{dV_3}{dt} =& (M - M_3)\left[rM(1 - \frac{M}{K}) - \frac{\alpha_1 MF}{1 + nG} + \frac{\beta G}{2} - \mu M\right]\\
&+ (F - F_3)\left[sF(1 - \frac{F}{K}) - \frac{\alpha_2 MF}{1 + nG} + \frac{\beta G}{2} - \mu F\right]\\
&+ \frac{\alpha_3 MF}{1 + nG} - (\beta + \gamma)G.
\end{aligned}
$$

Further simplification leads to:

$$
\begin{aligned}
\frac{dV_3}{dt} =& - q_{11}(M - M_3)^2 - q_{12}(M - M_3)(F - F_3) - q_{22}(F - F_3)^2\\
&+ q_1^\star(M - M_3) + q_2^\star(F - F_3) - q_3^\star G,
\end{aligned}
$$

where

$$
\begin{aligned}
q_{11} &= \frac{r}{K}(M + M_3) + \mu + \frac{\alpha_1}{1 + nG} - r,\\
q_{22} &= \frac{s}{K}(F + F_3) + \mu + \frac{\alpha_2 M_3}{1 + nG} - s,\\
q_{12} &= \frac{\alpha_1 M_3}{1 + nG} + \frac{\alpha_2 F}{1 + nG},\\
q_1^\star &= \left[\frac{n\alpha_1 M_3 F_3 G}{1 + nG} + \frac{\beta G}{2} + \frac{\alpha_3 F}{1 + nG}\right],\\
q_2^\star &= \left[\frac{n\alpha_2 M_3 F_3 G}{1 + nG} + \frac{\beta G}{2} + \frac{\alpha_3 M_3}{1 + nG}\right],\\
q_3^\star &= \left[\frac{n\alpha_3 M_3 F_3 G}{1 + nG} + \beta + \gamma\right].
\end{aligned}
$$

Thus, $dV_3/dt \leq 0$, under conditions (21). Hence, the MFEP is GAS. □

Theorem 4.8. *The MEP is globally asymptotically stable under the following requirement holds.*

$$
\begin{aligned}
\left[2 - \frac{M}{M_4} - \frac{M_4}{M}\right] &\leq 0,\\
\left[2 - \frac{F}{F_4} - \frac{F_4}{F}\right] &\leq 0,\\
\left[2 - \frac{G}{G_4} - \frac{G_4}{G}\right] &\leq 0.
\end{aligned} \tag{22}
$$

Proof. Let define the function $V_4(M,F,G) = \int_{M_4}^{M}(1-\frac{M_4}{x})dx + \int_{F_4}^{F}(1-\frac{F_4}{x})dx + \int_{G_4}^{G}(1-\frac{G_4}{x})dx$, which is a positive definite real valued function on the region $\Omega_4 = (M,F,G) \in R_+^3 : M > 0, F > 0, G > 0$. Then, after simplify it by some direct calculations, we have

$$\left(1-\frac{M_4}{M}\right)\frac{dM}{dt} = \mu M_4\left[2-\frac{M}{M_4}-\frac{M_4}{M}\right] + rM_4\left[2-\frac{M}{M_4}-\frac{M_4}{M}\right] + \frac{rM_4^2}{K}\left[1-\frac{M_4}{M}+\frac{M}{M_4}\left(1-\frac{M}{M_4}\right)\right] + \frac{\beta G_4}{2}\left[1-\frac{G}{G_4}+\frac{GM_4}{G_4M}\left(1-\frac{G_4}{G}\right)\right] + \frac{\alpha_1 M_4F_4}{1+nG_4}\left[1-\frac{M_4}{M}+\frac{F(1+nG_4)}{F_4(1+nG)}\left(1-\frac{M}{M_4}\right)\right],$$

$$\left(1-\frac{F_4}{F}\right)\frac{dF}{dt} = \mu F_4\left[2-\frac{F}{F_4}-\frac{F_4}{F}\right] + SF_4\left[2-\frac{F}{F_4}-\frac{F_4}{F}\right] + \frac{sF_4^2}{K}\left[1-\frac{F_4}{F}+\frac{F}{F_4}\left(1-\frac{F}{F_4}\right)\right] + \frac{\beta G_4}{2}\left[1-\frac{G}{G_4}+\frac{GF_4}{G_4F}\left(1-\frac{G_4}{G}\right)\right] + \frac{\alpha_2 M_4F_4}{1+nG_4}\left[1-\frac{F_4}{F}+\frac{M(1+nG_4)}{M_4(1+nG)}\left(1-\frac{F}{F_4}\right)\right],$$

$$\left(1-\frac{G_4}{G}\right)\frac{dG}{dt} = (\beta+\gamma)G_4\left[2-\frac{G}{G_4}-\frac{G_4}{G}\right] + \frac{\alpha_3 M_4F_4}{1+nG_4}\left[1-\frac{G_4}{G}+\frac{MFG_4(1+nG_4)}{M_4F_4G(1+nG)}\left(1-\frac{G_4}{G}\right)\right].$$

Thus, if $M = M_4$, $F = F_4$ and $G = G_4$, we get the $dV_4/dt \le 0$, under conditions (22). Hence, the MEP is GAS. □

5 Bifurcation Analysis of Model (1)

There are many types of bifurcation that occur as a result of change in the qualitative behaviour of the equilibrium points are explored in this section. We show that the model (1) goes through pitch-fork bifurcation, saddle-node bifurcation and transcritical bifurcation, all of which are local bifurcations in co-dimension one.

Theorem 5.1. *Model (1) has a saddle-node bifurcation near the W_4, (MEP), but neither transcritical bifurcation, nor pitchfork bifurcation for the parameter γ and the threshold value is $\gamma = \gamma^*$.*

Proof. The occurrence of a saddle-node bifurcation at the bifurcation parameter $\gamma = \gamma^*$ is confirmed by using Sotomayor's theorem [12]. Provided that the following conditions are met:

$$\begin{aligned} \gamma^* &> 0, \\ \varphi_3\left(l_1c_{11}^* + l_2c_{21}^* + c_{31}^*\right) &\neq 0, \end{aligned} \tag{23}$$

where

$$\gamma^* = \frac{d_{13}\left(d_{21}d_{32}-d_{22}d_{31}\right) + d_{23}\left(d_{12}d_{31}-d_{11}d_{32}\right)}{d_{11}d_{22}-d_{12}d_{21}} - \left(\frac{n\alpha_3M_4F_4}{1+nG_4}+\beta\right).$$

Now, the Jacobian matrix for model (1) about the equilibrium point (MEP) and $\gamma = \gamma^*$ has a zero eigenvalue, say $\lambda_4 = 0$ and as such, the W_4 becomes a non-hyperbolic point. Let $v = (k_1v_3, k_2v_3, v_3)^T$ and $\varphi = (l_1\varphi_3, l_2\varphi_3, \varphi_3)^T$ be eigenvectors of the Jacobian J_{W_4} and its transpose matrix $J^T_{W_4}$, corresponding to the zero eigenvalue λ_4, respectively, where

$$\begin{aligned} k_1 =& \frac{d_{12}(d_{11}d_{23} - d_{21}d_{13})}{d_{11}(d_{11}d_{22} - d_{12}d_{21})} + \frac{d_{13}}{d_{11}}, \\ k_2 =& \frac{d_{13}d_{21} - d_{11}d_{23}}{d_{11}d_{22} - d_{12}d_{21}}, \\ l_1 =& \frac{d_{21}(d_{11}d_{32} - d_{21}d_{31})}{d_{11}(d_{11}d_{22} - d_{12}d_{21})}, \\ l_2 =& \frac{d_{12}d_{31} - d_{11}d_{32}}{d_{11}d_{22} - d_{12}d_{21}}. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{\partial}{\partial \gamma} F(x, \gamma) =& (0, 0, -G)^T, \\ F_\gamma(W_4, \gamma^*) =& (0, 0, -G_4)^T \Rightarrow \varphi^T F_\gamma(W_4, \gamma^*) = -\varphi_3 G_4, \\ \varphi^T \left[D^2 F(W_4, \gamma^*)(v, v) \right] =& \varphi_3 \left(l_1 c^*_{11} + l_2 c^*_{21} + c^*_{31} \right), \end{aligned}$$

where

$$\begin{aligned} c^*_{11} =& -2v_3^2 \left[\frac{rk_1}{K} + \frac{\alpha_1 k_1 k_2}{1 + nG_4} + \frac{n\alpha_1 F_4 k_1}{(1 + nG_4)^2} + \frac{n\alpha_1 k_2 M_4}{(1 + nG_4)^2} \right], \\ c^*_{21} =& -2v_3^2 \left[\frac{\alpha_2 k_1 k_2}{1 + nG_4} + \frac{sk_2}{K} + \frac{n\alpha_2 k_1 F_4}{(1 + nG_4)^2} + \frac{n\alpha_2 k_2 M_4}{(1 + nG_4)^2} \right], \\ c^*_{31} =& 2v_3^2 \left[\frac{\alpha_3 k_1 k_2}{1 + nG_4} - \frac{n\alpha_3 k_1 k_2}{(1 + nG_4)^2} - \frac{n\alpha_3 k_2 M_4}{(1 + nG_4)^2} + \frac{n^2 \alpha_3 M_4 F_4}{(1 + nG_4)^2} \right]. \end{aligned}$$

If the conditions (23), are hold. Then, all transversality conditions are satisfied, i.e. when $\gamma = \gamma^*$, the saddle-node bifurcation occurs at W_4. □

Theorem 5.2. *Model (1) has a transcritical bifurcation near the W_3, (MFEP), but neither saddle-node bifurcation, nor pitchfork bifurcation for the parameter γ and the threshold value is $\gamma = \gamma^\star$.*

Proof. The occurrence of a transcritical bifurcation at the bifurcation parameter $\gamma = \gamma^\star$ is confirmed by using Sotomayor's theorem [12]. Provided that the following conditions are met:

$$\begin{aligned} \gamma^\star > 0, \\ \varphi^\star_1 \left(c^\star_{11} + l^\star_1 c^\star_{21} + l^\star_2 c^\star_{31} \right) \neq 0, \end{aligned} \tag{24}$$

where

$$\gamma^\star = \frac{c_{13} \left(c_{22}c_{31} - c_{21}c_{32} \right) + c_{23} \left(c_{11}c_{32} - c_{12}c_{31} \right)}{c_{12}c_{21} - c_{11}c_{22}} - \left(n\alpha_3 M_3 F_3 + \beta \right).$$

Now, the Jacobian matrix for model (1) about the equilibrium point (MFEP) and $\gamma = \gamma^\star$ has a zero eigenvalue, say $\lambda_3 = 0$ and as such, the W_3 becomes a non-hyperbolic point. Let $v^\star = (k^\star_1 v^\star_2, v^\star_2, k^\star_2 v^\star_2)^T$ and $\varphi^\star = (\varphi^\star_1, l^\star_1 \varphi^\star_1, l^\star_2 \varphi^\star_1)^T$ be eigenvectors of the Jacobian J_{W_3} and its transpose matrix $J_{W_3}^T$, corresponding to the zero eigenvalue λ_3, respectively, where

$$\begin{aligned} k_1^\star =& \frac{c_{13}c_{22} - c_{12}c_{23}}{c_{11}c_{23} - c_{21}c_{13}}, \\ k_2^\star =& \frac{c_{12}c_{21} - c_{11}c_{22}}{c_{11}c_{23} - c_{13}c_{21}}, \\ l_1^\star =& \frac{c_{31}c_{13} - c_{11}c_{33}}{c_{33}c_{21} - c_{23}c_{31}}, \\ l_2^\star =& \frac{c_{11}c_{23} - c_{13}c_{21}}{c_{33}c_{21} - c_{23}c_{31}}. \end{aligned} \tag{25}$$

Then, we have

$$\frac{\partial}{\partial \gamma} F(x,\gamma) =(0,0,-G)^T, \tag{26}$$

$$F_\gamma(W_3,\gamma^\star) =(0,0,0)^T \Rightarrow {\varphi^\star}^T F_\gamma(W_3,\gamma^\star) = 0, \tag{27}$$

$${\varphi^\star}^T DF_\gamma(W_3,\gamma^\star)v^\star =- k_2^\star l_2^\star \varphi_1^\star v_2^\star \neq 0, \tag{28}$$

$${\varphi^\star}^T \left[D^2F(W_3,\gamma^\star)(v^\star,v^\star)\right] =\varphi_1^\star\left(c_{11}^\star + l_1^\star c_{21}^\star + l_2^\star c_{31}^\star\right), \tag{29}$$

where

$$c_{11}^\star =- 2k_1^\star {v_2^\star}^2 \left[\frac{k_1^\star}{2}\left(\frac{2r}{K}+1\right) + \alpha_1 + \frac{\alpha_1 n^2 M_3 F_3 {k_2^\star}^2}{k_1^\star} - n\alpha_1 k_2^\star \left(F_3 + \frac{M_3}{k_1^\star}\right)\right], \tag{30}$$

$$c_{21}^\star =- 2k_1^\star l_1^\star {v_2^\star}^2 \left[\alpha_2 + \frac{\alpha_2 n^2 M_3 F_3 {k_2^\star}^2}{k_1^\star} + \frac{s}{Kk_1^\star} - n\alpha_2 k_2^\star \left(F_3 + \frac{M_3}{k_1^\star}\right)\right], \tag{31}$$

$$c_{31}^\star =- 2\alpha_3 k_1^\star l_2^\star {v_2^\star}^2 \left[nk_2^\star F_3 + \frac{nk_2^\star M_3}{k_1^\star} - \left(1 + \frac{n^2 {k_2^\star}^2 M_3 F_3}{k_1^\star}\right)\right]. \tag{32}$$

If the conditions (24), are hold. Then, all transversality conditions are satisfied, i.e. when $\gamma = \gamma^\star$, the transcritical bifurcation occurs at W_3. □

6 Numerical Simulation

Now, we perform some practical simulations by using Matlab software. The authors used number of sets to perform the graphs. In our simulations, we are used sets of parameters values are given in Table 2 from various sets of initial values.

Table 2: Values of the model parameters.

Parameter	Value
r	1
s	0.9
β	0.03
K	25
n	100
α_1	0.9
α_2	0.7
α_3	0.5
μ	0.01
γ	0.001

According to Figure 2, model (1) has a unique *MEP* in MFG-space, which is GAS.

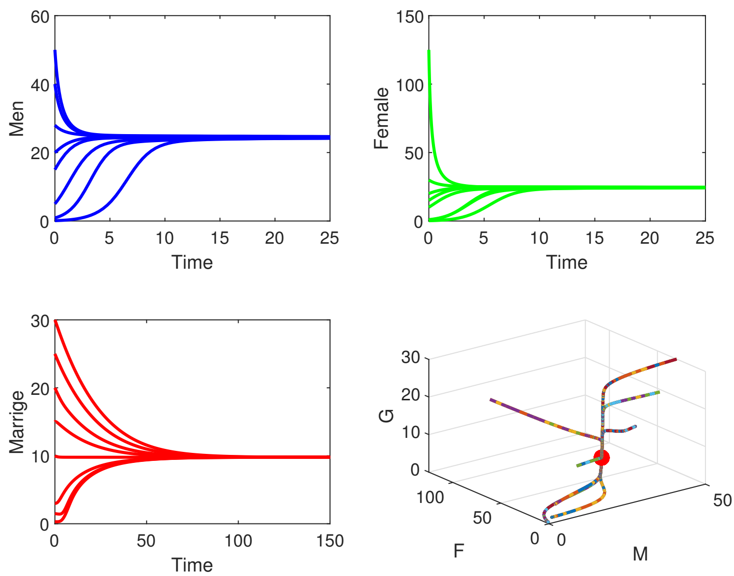

Figure 2: Simulation results of the global stability of the marriage equilibrium point.

Now, a numerical investigation of the effects of changing the parameters on the dynamics of model (1) is conducted. Thus, we get the following scenarios:

- Scenario 1: If we put $\alpha_1 = 0.1$ and $\alpha_3 = 0.001$ and keeping the other parameters value of Table 2. In this case, the dynamical behavior of model (1) converges to the *SMEP* from different initial values. This result is plotted by Figure 3.

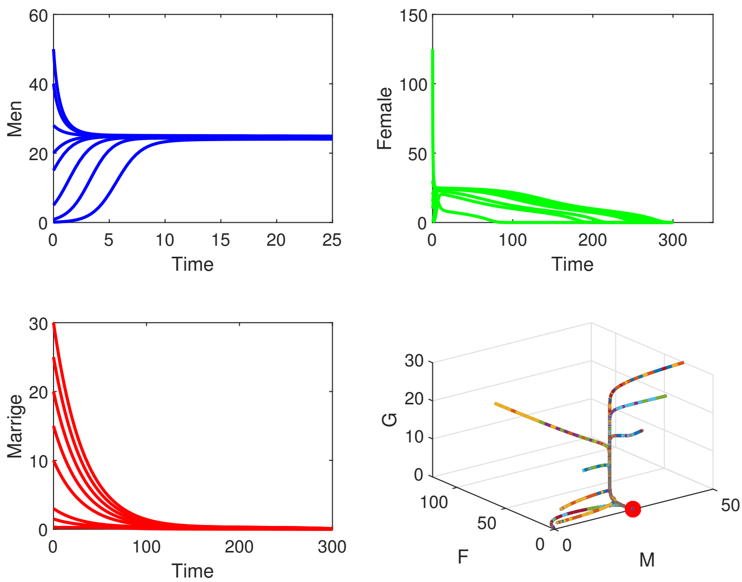

Figure 3: Simulation results of the global stability of the single males equilibrium point when $\alpha_1 = 0.1$ and $\alpha_3 = 0.001$.

- Scenario 2: If we put $\alpha_2 = 0.1$ and $\alpha_3 = 0.001$ and keeping the other parameters value of Table 2. In this case, the dynamical behavior of model (1) converges to the *SFEP* from different initial values. This result is plotted by Figure 4.

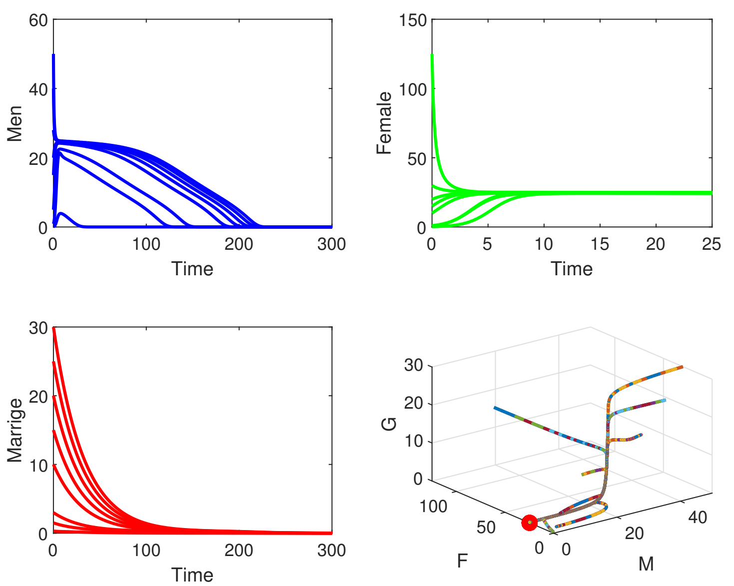

Figure 4: Simulation results of the global stability of the single females equilibrium point when $\alpha_2 = 0.1$ and $\alpha_3 = 0.001$.

- Scenario 3: If we put $\alpha_3 = 0.001$, $\beta = 0.8$ and $n = 8000$ and keeping the other parameters value of Table 2. In this case, the dynamical behavior of model (1) converges to the *MFEP* from different initial values. This result is plotted by Figure 5.

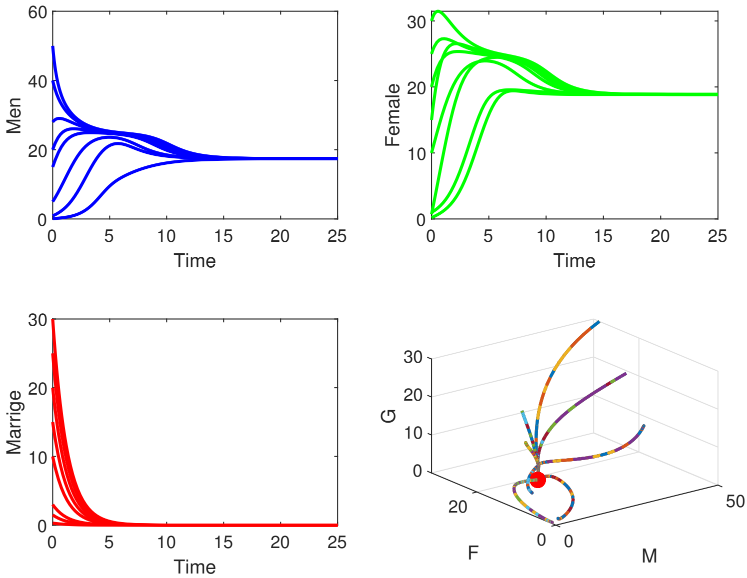

Figure 5: Simulation results of the global stability of the marriage-free equilibrium point when $\alpha_3 = 0.001$, $\beta = 0.8$ and $n = 8000$.

- Scenario 4: Obviously, Figure 6 shows clearly the bifurcation occur of the dynamical behavior of model (1) from the marriage equilibrium point W_4 became unstable to single females equilibrium point W_2 when the marriage rate α_3 is decreasing. While, we get the same results in Figure 6 if the decay rate γ is increasing see Figure 7.

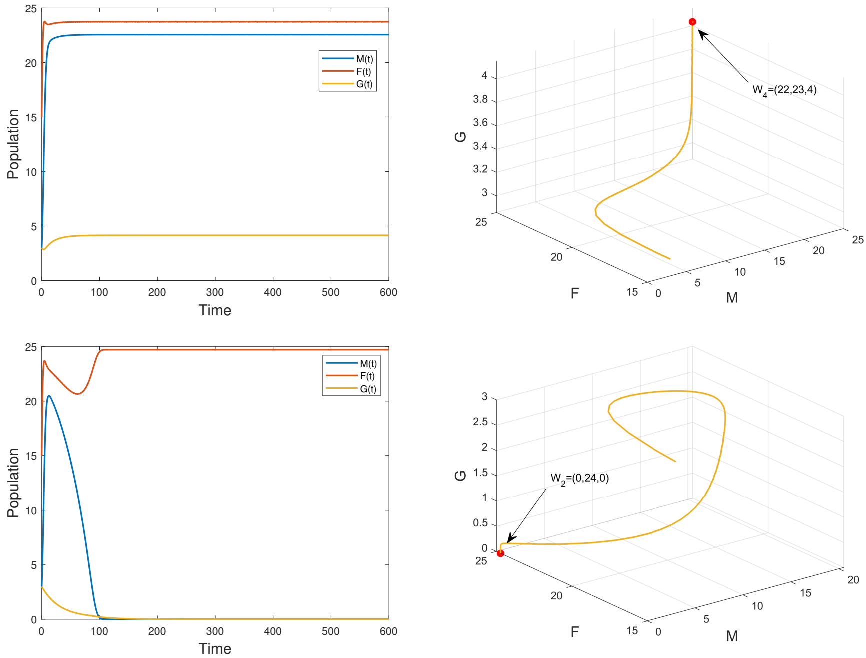

Figure 6: Simulation results of the dynamical behavior of model (1) when $\alpha_3 = 0.4, 0.002$ respectively.

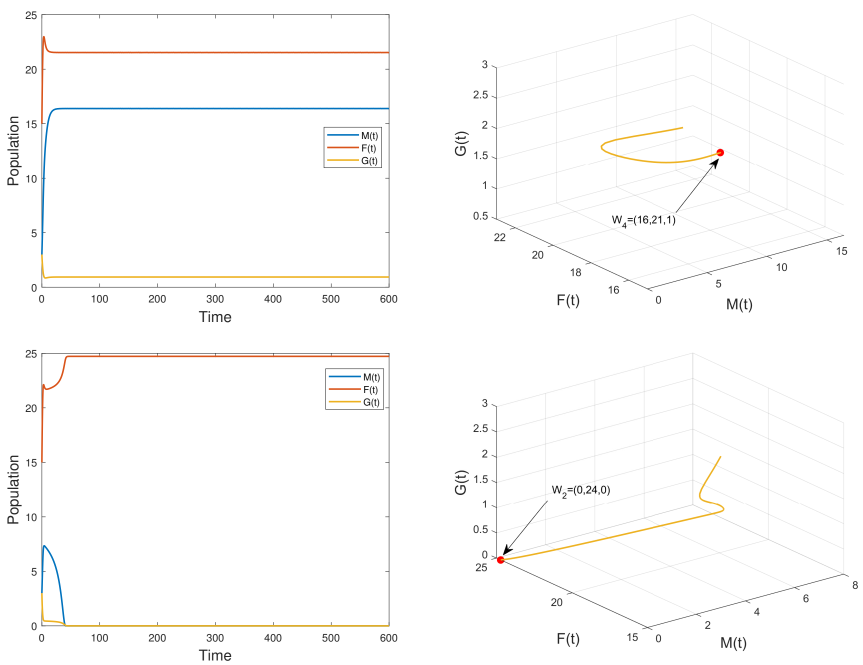

Figure 7: Simulation results of the dynamical behavior of model (1) when $\gamma = 0.1, 0.5$ respectively.

7 Conclusions

In this paper, we have proposed and studied a social model of the marriage relationships under fear effect with nonlinear incidence rate. We discussed the stability of the equilibrium points (the single males equilibrium point W_1, the single females equilibrium point W_2, the marriage-free equilibrium point W_3 and the marriage equilibrium point W_4). W_1 is locally asymptotically stable once the conditions 10 are hold and W_2 is locally asymptotically stable under the conditions 13 are hold as well as W_3 is locally asymptotically stable under the conditions 15 are hold, also, the last point W_4 is locally asymptotically stable if the conditions 17 are satisfied. Later, we studied the global stability of all equilibrium points of model (1) with help Lyapunov function. As the decay of the marriage rate γ is increasing through the equilibrium point W_4, the differential-algebraic model behavior goes to pitch-fork bifurcation and the stability of W_4 is lost. Due to the increase in divorce cases over time and thus the number of males decreases, this means that it W_2 is stable. Finally, some numerical simulations that support the obtained analytical results are given. The delay effect or fractional order remains an open problem for this model. This open problem will be investigated in our future research.

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Conflicts of Interest The authors declare no conflict of interest.

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